Auctioning vs. grandfathering in cap-and-trade systems with market power and imperfect information

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Abstract

We present a model with a cap-and-trade system in which there is imperfect competition and some firm has market power. In this framework, we study the different implications, in terms of efficiency, of the system used for the initial allocation of permits, either an auction or grandfathering. We show that, if the firm with market power in the secondary market also holds such a power in the auction, then the auction always generates more abatement costs than a grandfathering system, but this is not the case if both firms act non-strategically in the auction. The main message is that the auction design matters in the sense that the chances that an auction may improve the grandfathering results or not crucially depends on whether the market power spills over to the auction or not.

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1 Introduction

As noted by Hepburn et al. (2006) most economists tend to recommend more auctioning while business tend to oppose to it. In the case of emission trading systems, and despite all the academic recommendations, auctioning has been the exception rather than the rule until very recently, while grandfathering has been, by far, the most widespread method used to distribute emission permits. Nevertheless, the situation is beginning to change and it could change even more dramatically in the coming future. In the European
Union Emission Trading System (EU ETS), considered to be largest environmental market in the world, the role of auctioning is becoming very relevant. Indeed, the 2008 revision of the European Emission Trading Directive foresees as a fundamental change for the third trading period, starting in 2013, that auctioning of allowances will be the rule rather than the exception.

From 2013 all allowances not allocated free of charge must be auctioned and auctioning will progressively replace free allocation as the main method in all EU ETS sectors except aviation. It is expected that roughly half of the allowances will be auctioned. This is in sharp contrast to the first trading period (2005-2007) in which a 5% limit was set to the amount of allowances that could be auctioned. Moreover, only four countries used auctions at all and only Denmark used the 5% limit. The situation in the second trading period (2008-2012) has not been very different, with no more than 4% of all the allowances auctioned. The arguments posed by the European Commission to support the introduction of auctions in the third period are that auctioning will ensure the efficiency, transparency and simplicity of the system, will create more incentives for abatement investments and will eliminate windfall profits1.

The world first CO$_2$ cap-and-trade system to require the widespread use of auctions is the Regional Greenhouse Gas Initiative (RGGI). This programme began in 2009, it includes the 10 northeastern US states as well as Eastern Canada and it covers CO$_2$ emissions from electricity generators. As noted by Brutraw et al (2009), the RGGI proposal represents a substantial break with the past since, instead of giving the permits away for free, the RGGI states decided to auction close to 90 percent of their permit budgets. The prominent role that auctioning is getting to play in these important examples introduces an important scientific motivation to analyze carefully the role of auctions in allocating emission permits.

We focus on a specific aspect of cap-and-trade systems which is cost-effectiveness and we try to answer a simple question: How do auctioning does as compared to grandfathering in terms of cost efficiency when there exists market power in the secondary market. As far as we know, this question has not been addressed in formal terms, although some related informal discussion and experimental studies have been conducted (see Ledyard and Szakaly-Moore (1994), Godby (1999, 2000), Muller et al, 2002). One exception is Antelo and Bru (2009) that focuses on an emissions permit market with a dominant firm in which the government is concerned both about social efficiency and permits revenue.

Our paper is somehow related to Antelo and Bru (2009), but it differs, first, in that we disregard the revenue dimension to focus just on cost-effectiveness and, second, in that we do a more exhaustive analysis of the auction properties. Notably, as it is common in auction theory, we introduce random types to capture the fact that cap-and-trade systems are typically designed in an uncertain environment whereas Antelo and Bru use a purely deterministic model. A similar approach is followed by Montero (2009). As a consequence, both Antelo and Bru (2009) and Montero (2009) conclude that, in a setting in which the

1 See http://ec.europa.eu/clima/policies/ets/auctioning/
auction is followed by a secondary market, it is optimal for the leader not to take part in the auction and acquire all the permits in the secondary market. We show that this is not necessarily the case under an imperfect observation framework. The certainty assumption also has an important implication for grandfathering. Whenever the planner has perfect information about the abatement cost functions, such knowledge can be used to implement perfectly the cost-effective allocation. In a more realistic setting, it can be argued that the planner has no perfect information about the abatement costs, what makes the efficient solution to be non-implementable in practice.

As it is well known in the emissions permits literature, whenever there is a perfectly competitive secondary market and full information, the initial distribution of permits only matters for the distribution of the gains of trade, but it is irrelevant for the sake of efficiency since the interaction of polluters will lead the market equilibrium to the most cost-effective solution whatever the initial allocation is (see, for example, Tietenberg 2006). Nevertheless, as first noted by Hahn (1984), if there is market power in the secondary market, the initial distribution of permits does matter for the sake of efficiency. Indeed, the cost-effective allocation will be reached only when, in the initial allocation, the leader receives exactly its emission under perfectly competitive pricing.

Sturn (2007) notes that whether market power actually constitutes a problem has to be analyzed for each emissions trading market separately. Market power is probably not one of the main issues in the EU ETS since the number of involved facilities is very large, but it may constitute a serious problem for future international emissions trading system within the framework of the Kyoto Protocol and for some regional emissions markets. Montero (2009) and Muller et al (2002) stress that market power issues are more likely to show up in markets where countries -rather than individual facilities- are the relevant players, as is the case of the Kyoto Protocol. Moreover it can be argued that, even in cap-and-trade systems with a large number of small pollutant firms, it is not unrealistic to assume that the secondary market will not be typically perfectly competitive since there might be information asymmetries, collusive behavior and other market failures.

We present a simple model in which two firms receive an initial allocation of pollution permits (either by means of grandfathering or an auction) and then interact in a secondary market in which one of the firms acts as a leader and the other one as a follower. We compare auctioning and grandfathering in terms of efficiency when used as an initial allocation method for emission permits. Regarding the auction, we consider, first, a framework in which the market power that the leader enjoys in the secondary market is reflected in the auction and, second, a case in which both firms behave non-strategically in the auction. For simplicity, we abstract from the goods market to focus on the market for permits.

Our preliminary results show that, under certain circumstances, the auction renders an initial permit allocation from which the total cost of abatement is higher than what would result from grandfathering with
probability one. On the contrary, if both firms act non-strategically, there is a strictly positive probability that the auction generates lower abatement cost than grandfathering. The main message that we can draw from these result is that the auction design is crucial for the possibility to improve grandfathering in terms of efficiency. Indeed, the changes that an auction may improve the grandfathering results or not crucially depends on whether the market power spills over to the auction or not.

The reminder of the paper proceeds as follows. In section 2 we present the basic elements of the model. In section 3 we study a version of the auction in which the leader acts strategically where the follower does not. In section 4 we present a different case in which both firms act non-strategically. Section 5 concludes. The formal proofs of the results are gathered in an appendix at the end of the paper.

2 The model

There are two polluting firms labelled as $L$ and $F$ (where $L$ stands for "leader" and $F$ for "follower"), that are subject to a tradeable emission permit system. $Q$ emission permits are issued and distributed between the firms by some allocation procedure (either an auction or grandfathering). Each firm receives an initial allocation of permits denoted as $q_{L0}$ and $q_{F0}$ respectively, where $q_{L0} + q_{F0} = Q$, and then the permits can be traded in a secondary market in which $L$ acts as a leader and $F$ as a follower.

To meet the environmental legal requirements, the firms have the options, first, to use permits and, second, to do some abatement effort to reduce their emissions. The latter option entails an abatement cost for each firm determined by the (total) cost functions $TC_L(e_L, \alpha_L)$ and $TC_F(e_F, \alpha_F)$ respectively, where $e_i$ are the effective emissions of firm $i$ ($i = L, F$) and $\alpha_i$ are random variables that, as is usual in auction theory, we call "types". These types capture the idea that abatement cost is subject to some uncertainty due to climatic, environmental or technological reasons. We assume that the distribution function of both types is common knowledge. Moreover, at the beginning of the game each firms observes its own type but only knows the distribution function of the rival’s type. After the initial allocation is made, the types are revealed and then firms engage in the secondary market and trade allowances with full information.

If we denote as $q_{i1}$ the final amount of permits that firm $i$ ($i = L, F$) holds after negotiation in the secondary market (that has to coincide with its realized emissions), the amount of permits sold (if $q_{i0} > q_{i1}$) or bought (if $q_{i1} > q_{i0}$) in the secondary market is given by $(q_{i0} - q_{i1})$. In the secondary market, the aim of both firms is to maximize their profit, what includes the revenue or the expenses due to permits trading, plus abatement costs:

$$\pi_L = p_L(q_{L0} - q_{L1}) - TC_L(q_{L1}, \alpha_L)$$
$$\pi_F = p_F(q_{F0} - q_{F1}) - TC_F(q_{F1}, \alpha_F)$$
where \( p_1 \) is the price of permits in the secondary market. The difference between both firms’ behavior is that the follower chooses its net demand for permits, \( q_{F1} \), (which is equivalent to choosing its emissions) while taking \( p_1 \) as given, whereas the leader chooses its net demand for permits taking into account the effect of its behavior on the market price. Therefore, the follower’s inverse demand for permits is given by the first order condition

\[
p_1 = MC_F (q_{F1}, \alpha_F)
\]

where \( MC_F (q_{F1}, \alpha_F) = -\frac{\partial TC_F (q_{F1}, \alpha_F)}{\partial q_{F1}} \) is the follower’s marginal cost of abatement. The follower maximizes \( \pi_F \) subject to the market clearing condition \( q_{L1} + q_{F1} = Q \), what results in the first order condition

\[
[MC_L (q_{L1}, \alpha_L) - p_1] \frac{\partial q_{F1}}{\partial p_1} + (Q - q_{F1} - q_{L0}) = 0
\]

and, as noted by Hahn (1984), the only case in which the marginal abatement cost of the leader will equal the equilibrium price is when, in the initial allocation, the leader receives exactly the amount of permits that corresponds to the cost-effective solution.

For our purposes, the main implication of this fact is that the initial allocation of permits is crucial for the sake of cost-effectiveness. If the planner had perfect information (in our framework, if the values of \( \alpha_F \) and \( \alpha_L \) were perfectly known), then it would be possible to compute the cost-minimizing solution and, by allocating the associated amounts of permits to the firms by means of grandfathering, cost-effectiveness would be achieved with certainty (and the secondary market would be redundant). Such a conclusion is obtained, for example, by Antelo and Bru (2009) in a framework of certainty.

Nevertheless, it is reasonable to think that the planner does not have perfect information since the allocation of permits is not done ad-hoc once all the realizations of the types are known. On the contrary, the allocation is performed by means of a mechanism, either grandfathering or auctioning, that is designed a priori, to be applied for any realization of the types before such realizations are observed.

Grandfathering consists in allocating permits for free based on historical emissions. Since abatement costs are random, and therefore, so are emissions, we assume that the planner uses the average of observed emissions as a reference for the allocation (in terms of our model, the permits are allocated based on the expected business-as-usual emissions). The alternative method is to conduct an auction in which the firms can bid the permits. Details on the auction will be discussed below.

For the sake of tractability, during the rest of the paper, we will assume linear functions of marginal abatement cost, what implies a quadratic total cost function:

\[
MC_i = \alpha_i - \beta e_i \\
TC_i = \int_{e_i}^{e_i^{BU}} (\alpha_i - \beta t) dt = \frac{\alpha_i^2}{2\beta} - \alpha_i e_i + \frac{\beta e_i^2}{2}, \quad i = L, F
\]
where \( e_{i}^{BAU} = \frac{\alpha_i}{\beta} \) are the business-as-usual (BAU) emissions of firm \( i \) \((i = L, F)\), i.e., the emissions that would prevail if no environmental policy were applied at all.\(^2\) For the sake of comparison it is useful to compute the efficient permit allocation\(^3\) \((e_{L}^{*}, e_{F}^{*})\), which follows from equating marginal costs of abatement between both firms:

\[
e_{L}^{*} = \frac{\alpha_{L} - \alpha_{F} + 2Q}{3\beta}, \quad e_{F}^{*} = \frac{\alpha_{F} - \alpha_{L} + 2Q}{3\beta}.
\] (1)

It is immediate to note that, when both firms are identical \((\alpha_{F} = \alpha_{L})\), it is optimal that each firm gets the same amount of permits, \(\frac{Q}{2}\).

### 2.1 The secondary market

Once the amounts of permits \(q_{L0}\) and \(q_{F0}\) have been allocated to firms \(L\) and \(F\) respectively (regardless of the procedure used for this allocation) both firms can trade permits in the secondary market. To solve this part of the game, we start by deriving the optimal behavior of the follower. Since \(F\) acts as a price taker, it solves the following problem

\[
\max_{\{q_{F1}\}} \pi_{F} = p_{1}(q_{F0} - q_{F1}) - \left[\frac{\alpha_{F}^2}{2\beta} - \alpha_{F}q_{F1} + \frac{\beta}{2}q_{F1}^2\right]
\]

that gives, as a result, the net demand function of permits for the follower,

\[
q_{F1} = \frac{1}{\beta}(\alpha_{F} - p_{1})
\]

which, as expected, is decreasing in the price of permits and increasing in the realized value of the follower’s type, \(\alpha_{F}\), but is independent of the initial allocation of permits.

The leader, in turn, decides its optimal demand for permits taking into account the effect of its own strategy on the market as well as the follower’s demand and the market clearance condition, i.e.,

\[
\max_{\{q_{L1}\}} \pi_{L} = p_{1}(q_{L0} - q_{L1}) - \left[\frac{\alpha_{L}^2}{2\beta} - \alpha_{L}q_{L1} + \frac{\beta}{2}q_{L1}^2\right]
\]

\[
s.t. \quad q_{L1} + \frac{1}{\beta}(\alpha_{F} - p_{1}) = Q
\]

from which we get the optimal leader’s demand, given by

\[
q_{L1} = \frac{\alpha_{L} - \alpha_{F}}{3\beta} + \frac{Q + q_{L0}}{3}
\]

\(^2\)The BAU emissions are computed as the value of \(e_{i}\) such that \(MC_{i} = 0\).
\(^3\)Stricto senso, \((e_{L}^{*}, e_{F}^{*})\) is what is normally called a cost-effective solution, since we are only considering abatement costs and not the social damage of pollution. Since we are not dealing with the issue of how to determine the total number of permits, it is innocuous to assume that this number has been chosen to minimize the external cost of pollution plus the abatement cost and, therefore, efficiency and cost-effectiveness turn out to be equivalent.
and using the market clearing condition and the demand function for $F$, we get the equilibrium price in the secondary market and the equilibrium number of permits for the follower:

$$p_1 = \frac{\alpha_L + 2\alpha_F}{3} - \frac{\beta (2Q - q_{L0})}{3}$$

$$q_{F1} = \frac{\alpha_F - \alpha_L + \beta (2Q - q_{L0})}{3\beta}$$

and, again, it is important to notice that the equilibrium reached in the secondary market (including the price and the final distribution of permits) depends on the amount of permits initially allocated to the leader. In particular, it is easy to check that, if $q_{L0} < e^*_L \,(q_L > e^*_L)$, then $e^*_L > q_{L1} > q_{L0}$ ($e^*_L < q_{L1} < q_{L0}$), i.e., if the leader receives initial less permits than in the cost-effective solution, he will buy permits in the secondary market acting as a monopsonist and will buy less permits than what would be socially desirable. If, on the contrary, he receives more permits than $e^*_L$, he will act as a monopolist and will sell less permits than what would be socially desirable.

Note that total abatement costs can be expressed as a function of the amount of permits that are originally allocated to the leader, $q_{L0}$. Indeed, using the expressions for $q_{L1}$ and $q_{F1}$ we can compute the aggregate value of total abatement costs as a function of the number of permits initially allocated to the leader:

$$\mathcal{T} \equiv \mathcal{T}_L + \mathcal{T}_F = \Theta + \frac{\alpha_F - \alpha_L - \beta Q}{9} q_{L0} + \frac{\beta}{9} q_{L0}^2$$

where $\Theta \equiv \frac{(5\alpha_F + 3\alpha_L - 5\beta Q)(\alpha_F + \alpha_L - \beta Q)}{18\beta}$ is a combination of parameters independent of the permits allocation. Therefore, using (2), the cost-effectiveness of a given allocation system can be computed directly from the amount of permits that are initially allocated to the leader since the allocative effect of the secondary market is already incorporated in (2). Note that minimizing (2) gives the cost-effective allocation shown in (1). One immediate conclusion that can be drawn from this analysis is that the cost-effectiveness of an allocation system can be assessed just by checking how close is the number of permits allocated to the leader under such system to the efficient amount $e^*_L$.

### 2.2 Grandfathering

To model the initial allocation of permits by means of grandfathering, assume that, before the tradeable permits system is implemented, the firms were not subject to any pollution control and, then, they emitted the amount of permits required for minimizing total abatement costs, i.e., the laissez-faire or BAU emissions, given by $e^{BAU}_i = \frac{\alpha_i}{\beta} \,(i = L, F)$. The main fact to be noted here is that, since $\alpha_i$ is a random variable, so are the BAU emissions. In our model, grandfathering consists in allocating permits based on historical (BAU) emissions but, since BAU emissions are random, the planner will not have a certain data

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to be used as a reference for the allocation. We assume that the allocation is based on average historical emissions or, in terms of our model, on the expected value of $e_i^{BAU}$, given by $e_i^{BAU} = \frac{\alpha_i}{\beta}$, where $\alpha_i$ is the expected value of $\alpha_i$.

Since the total amount of permits to be distributed is given and equal to $Q$, we assume that both firms receive an initial endowment of permits given by a fraction of $Q$ calculated proportionately to expected BAU emissions, i.e.,

$$q^G = \frac{e_i^{BAU}}{e_i^{BAU} + e_F^{BAU}}Q = \frac{\alpha_L}{\alpha_L + \alpha_F}Q, \quad q^F = \frac{\alpha_F}{\alpha_L + \alpha_F}Q$$

where "G" stands for grandfathering. And, plugging the value of $q^G$ in (2), we can compute the resulting total abatement costs that will be realized after the firms have interacted in the secondary market

$$TC^G = \Theta + \frac{\gamma_L (\alpha_F - \alpha_L - \beta Q) Q}{9} + \frac{\beta}{9} (\gamma_L)^2 Q^2$$

where $\gamma_L \equiv \frac{\alpha_L}{\alpha_L + \alpha_F}$ and $\gamma_F \equiv \frac{\alpha_F}{\alpha_L + \alpha_F}$ are the share of $L$ and $F$ in the total amount of permits issued.

3 Auction with strategic leader and non-strategic follower

The second option to make an initial allocation of permits is to conduct an auction. Since there are more than one permit to be distributed, it is a multi-unit auction. Since the number is typically very large, it seems a reasonable approximation to take the number of permits as a continuous, perfectly divisible good. Moreover, based on the format chosen by the European Union, we focus on a sealed-bid uniform auction, in which all the permits are sold at the same clearing price. We assume that the strategy space for each firm $i$ ($i = L, F$) is the space of all the functions $\tau_{i0}(p_0)$, where $p_0$ is the auction price. The equilibrium concept used is the Bayesian Nash equilibrium.

In this section we analyze the allocation impact of an auction in which the leader acts strategically in the sense that, when bidding, he takes into account the effect of his own bid on the clearing price of the auction while the follower acts non-strategically and takes the price as given. This case is based on the belief that, if one firm has some market power in the secondary market, due to its size or dominant position, it is likely that this power has a reflection in the auction. To capture this notion, we assume that the follower feels that it cannot influence the auction price in any way and, therefore, acts as a price-taker, whereas the leader is capable of predicting the follower’s strategy and devise its own strategy considering all the relevant effects on the equilibrium outcome of the auction.

The first step to solve the auction is to compute the (expected) value of profits in the secondary market

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4 See Wang and Zender (2002) for an analysis of divisible goods auctions.
for both firms. Using the expressions computed in section 2.1 we get the secondary market profits once the optimal strategies of the leader and the follower have been taken into account:

\[ \hat{\pi}_L^A (q_{L0}) = \Theta_L + \frac{\alpha_L + 2\alpha_F - 2\beta Q}{3} q_{L0} + \frac{\beta}{6} q_{L0}^2 \]
\[ \hat{\pi}_F^A (q_{F0}) = \Theta_F + \frac{2\alpha_L + 7\alpha_F - 2\beta Q}{9} q_{F0} - \frac{5\beta}{18} q_{F0}^2 \]

where \( \Theta_L \equiv -6(\alpha_F + \alpha_L)\alpha_L + \alpha_L^2 + 3\beta Q(2\alpha_L - \alpha_F + \beta Q) \), \( \Theta_F \equiv \frac{(\alpha_F - \alpha_L)\beta Q - 3\alpha_F^2 - 14\alpha_F^2 + 6\alpha_F \alpha_L + 3\beta Q \alpha_F^2}{18\beta} \) are two terms that depend on the parameters of the model as well as the types but are independent of the initial allocation (and, therefore, are irrelevant to design the bidding strategy of each player). "A" stands for "auction".

3.1 The follower’s strategy

We assume that the follower sees himself as being too small to influence the equilibrium price of the auction and, so, acts as a price-taker. In this situation, the best he can do is to submit a function \( \tau_F(p_0) \) with the aim to maximize his expected profit while taking \( p_0 \) as given. We define as \( V_F (q_{F0}, \alpha_F) \equiv E[\hat{\pi}_F(q_{F0}, \alpha_F)] \) the expected value of the follower’s profit in the secondary market conditional on the number of permits received in the auction and the realization of his own type. Then, the follower solves the following optimization problem:

\[
\max_{\{q_{F0}\}} V_F (q_{F0}, \alpha_F) - p_0 q_{F0}
\]

the first condition of which is \( \frac{\partial V_F (q_{F0}, \alpha_F)}{\partial q_{F0}} = p_0 \), that is equivalent to

\[
2E \{\alpha_L/\alpha_F\} + 7\alpha_F - 2\beta Q - 5\beta q_{F0} = p_0
\]

(4)

where \( E(\alpha_L/\alpha_F) \) is the expected value of \( \alpha_L \) conditional on the value of \( \alpha_F \), or, in other words, the best prediction that \( F \) can make about \( \alpha_L \) using his information set and it is easy to check that the second order necessary condition holds since the coefficient associated to \( q_{F0}^2 \) in \( \pi_F \) is negative. Solving (4) for \( q_{L0} \) we get optimal bidding function for the follower:

\[
\tau_F (p_0, \alpha_F) = \frac{7\alpha_F + 2E \{\alpha_L/\alpha_F\} - 2\beta Q - 9p_0}{5\beta}
\]

(5)

which is linear function with a rather intuitive behavior. First, note that \( \tau (p_0, \alpha_F) \) is decreasing in the price, as expected. Second, it is increasing both in \( \alpha_F \) and the expected value of \( \alpha_L \). Since increasing \( \alpha_F \) shifts the follower’s abatement cost upwards, it is reasonable that the higher \( \alpha_F \) the more \( F \) is willing to
pay for the permits. Regarding $\alpha_L$, although this parameter affects the leader’s, and not the follower’s cost, $F$ will forecast that a higher value of $\alpha_L$ will make the permits more valuable in the secondary market, what would make him be also more willing to pay for the same amount of permits. Finally, $\tau(p_0, \alpha_F)$ depends negatively on the total amount of issued permits, $Q$, since more issue permits will imply that it will be easier to get cheaper permits in the secondary market.

In what follows, to get a closed solution for the auction, we will hold the following assumption about the join distribution of $\alpha_L$ and $\alpha_F$:

**Assumption 1:** The expected value of $\alpha_L$ conditional on the value of $\alpha_F$ is given by

$$E \{ \alpha_L / \alpha_F \} = \lambda_0 + \lambda_1 \alpha_F$$

where $\lambda_0$ and $\lambda_1$ are given parameters. ■

Using Assumption 1 in (5) we get the following optimal bidding function for the follower:

$$\tau_F(p_0, \alpha_F) = \frac{2\lambda_0 + (2\lambda_1 + 7) \alpha_F}{5\beta} - \frac{2Q}{5} - \frac{9}{5\beta} p_0$$

(7)

### 3.2 The leader’s strategy

The leader acts strategically in the sense that he is capable of predicting the follower’s strategy and the effect of his own bid on the equilibrium price. Hence he solves the following problem:

$$\max_{\{q_{L0}\}} V_L (q_{L0}, \alpha_L) - p_0 q_{L0}$$

subject to $Q = q_{L0} + \tau(p_0, \alpha_F)$ and (7)

where $V_L (q_{L0}, \alpha_L) \equiv E[\pi_L / q_{L0}, \alpha_L]$. One important property of this problem is that the leader turns out to be capable of designing an ex-post optimal bid, i.e., a bid that gives him the maximum possible profit for any value of the types $\alpha_L$ and $\alpha_F$. To prove this result, the first step is to obtain the price that maximizes the leader’s profit. Consider the following artificial problem, which is written as if the price of the auction were a decision variable of the leader and the could know the real value of $\alpha_F$:

$$\max_{\{p_0\}} \pi_L \left( Q - \underset{\alpha_L}{\tau_F (p_0, \alpha_F); \alpha_F, \alpha_L} \right) - p_0 \times \left[ Q - \tau_F (p_0, \alpha_F) \right]$$

(8)

$\tau_F (p_0, \alpha_F)$ being given by (7).
Lemma 1 The optimal solution of problem (8) is given by

\[ p_0 = \frac{4\lambda_0}{63} + \frac{2(2\lambda_1 + 22)\alpha_F}{63} + \frac{15\alpha_L}{63} + \frac{44}{63}Q \]  

Lemma 1 shows that, for any values of the parameters and the types, it is possible to find a price that maximizes the leader’s profit. Proposition 2 shows that there exist a bidding function for the leader such that, the equilibrium price of the auction will be given exactly by (9) or, in other words, such that the result of the auction will be ex-post optimal for him given any values of the types.

Proposition 2 For any values of the parameters such that the equilibrium of the auction is interior, there exists a bidding function for the leader, given by

\[ \tau_L (p_0, y) = m_0 + m_1Q + m_2\alpha_L + m_3p_0 \]

where \( m_0 = \frac{-3\lambda_1}{\lambda_1 + 1} \), \( m_1 = \frac{-3\lambda_1}{\lambda_1 + 1} \), \( m_2 = \frac{6\lambda_1 + 21}{4\lambda_1 + 11} \), \( m_3 = \frac{-18\lambda_1 + 9}{4\lambda_1 + 11} \), such that the equilibrium price of the auction for any value of \( \alpha_F \) is given by (9).

Lemma 1 and Proposition 2 imply that the leader has the possibility to bid in such a way that, for any realization of the types, the result of the auction will be optimal for him. Note that \( \frac{\partial \tau_L}{\partial p_0} = m_3 < 0 \) if and only if \( \lambda_1 > -\frac{1}{2} \), i.e., the optimal bidding function submitted by the leader has negative slope with respect to the price provided that \( \lambda_1 \), the slope of the expected value of \( \alpha_L \) with respect to \( \alpha_F \), is not lower than \(-1/2\), what includes the cases in which both variables are uncorrelated \( (\lambda_1 = 0) \), positively correlated or slightly negatively correlated. The most counterintuitive case is that in which \( \lambda_1 < 0 \), what implies that the leader’s bidding function is increasing with respect to the price.

Plugging the value of the clearing price in the bidding function, we find the equilibrium number of permits in the auction for both firms:

\[ q_{L0}^A = \frac{\beta Q + 3\alpha_L - 2\lambda_0 - (2\lambda_1 + 1)\alpha_F}{\beta} \quad q_{F0}^A = \frac{2\lambda_0 + (2\lambda_1 + 1)\alpha_F + 6\beta Q - 3\alpha_L}{\beta} \]

And, using (2), we obtain the expression for total abatement costs under the auction:

\[ TC^A = \Theta + \frac{\beta Q + 3\alpha_L - 2\lambda_0 - (2\lambda_1 + 1)\alpha_F}{44\beta} [-6\beta Q - 4\alpha_L - 2\lambda_0 + (6 - 2\lambda_1)\alpha_F]. \]

3.3 Comparing auctioning and grandfathering under uniform distribution

To determine which allocation system is more cost-effective, we introduce the additional assumption that the types, \( \alpha_L \) and \( \alpha_F \), are jointly distributed according to a uniform distribution and, moreover, both
types are bounded between \( \theta \) and \( \theta + \sigma \), where \( \theta \) and \( \sigma \) are parameters that capture the size and the variability of the types respectively.

For simplicity, in this section we will focus on interior solutions in the sense that the equilibrium prices are positive and both firms receive positive amounts of permits both in the primary and the secondary market. Using the analytical expressions for the quantities and the prices, and rearranging, we get the following conditions:

\[
-13\beta Q \leq 2\lambda_0 + (2\lambda_1 + 8) \alpha_F - 10\alpha_L \leq 8\beta Q \quad (11)
\]
\[
-6\beta Q \leq 2\lambda_0 + (2\lambda_1 + 1) \alpha_F - 3\alpha_L \leq \beta Q \quad (12)
\]
\[
\beta Q \leq \frac{10\alpha_L + (13 - 2\lambda_1) \alpha_F - 2\lambda_0}{13} \quad (13)
\]
\[
\beta Q \leq \frac{\lambda_0 + (\lambda_1 + 11) \alpha_F}{11} + \frac{15\alpha_L}{44} \quad (14)
\]

The first condition guarantees interior solution for quantities in the secondary market (i.e., \( 0 < q_L^b, q_F^b < Q \)) and the second does the same for the auction (\( 0 < q_L^a, q_F^a < Q \)). The third and fourth conditions ensure that the price is positive in the secondary market and the auction respectively (\( p_1 > 0, p_0 > 0 \)).

For the sake of completeness, we consider three possible cases that are illustrated in Figure 1. In case 1, \( \alpha_L \) and \( \alpha_F \) are not correlated and so the distribution is uniformly defined over the square interval \([\theta, \theta + \sigma] \times [\theta, \theta + \sigma]\). In case 2, apart from the uniform distribution assumption, we assume that \( \alpha_L \leq \alpha_F \). As a result, the interval over which the distribution is defined is a triangle delimited by the points \((\theta, \theta), (\theta + \sigma, \theta)\) and \((\theta + \sigma, \theta + \sigma)\). Symmetrically, in case 3, we impose the constraint \( \alpha_L \geq \alpha_F \), what implies that the distribution is uniformly defined over a triangle delimited by the points \((\theta, \theta), (\theta, \theta + \sigma)\) and \((\theta + \sigma, \theta + \sigma)\). In the figure it is also illustrated the shape of the regression line that renders the expected value of \( \alpha_L \) conditional on the realization of \( \alpha_F \), \( E\{\alpha_L/\alpha_F\} \), under the relevant assumptions.

CASE 1: \( \alpha_L, \alpha_F \) uncorrelated

CASE 2: \( \alpha_L \leq \alpha_F \)

CASE 3: \( \alpha_L \geq \alpha_F \)

Figure 1. Intervals for the distribution of types
CASE 1: \( \alpha_L \) and \( \alpha_F \) are not correlated

As it is illustrated in Figure 1, when \( \alpha_L \) and \( \alpha_F \) are not correlated, the conditional expectation of \( \alpha_L \) is independent of the realization of \( \alpha_F \) and equal to the unconditional mean, \( \theta \). In other words, \( E\{\alpha_L/\alpha_F\} = \lambda_0 + \lambda_1 \alpha_F = \theta + \frac{\bar{\gamma}}{2} \) for any value of \( \alpha_F \). This allows as to determine that, in this case, it must be \( \lambda_0 = \theta + \frac{\bar{\gamma}}{2}, \lambda_1 = 0 \). When we use these values on the interior solution conditions (11)-(14), we get

\[
-13\beta Q \leq 2\theta + \sigma + 8\alpha_F - 10\alpha_L \leq 8\beta Q \tag{15}
\]
\[
-6\beta Q \leq 2\theta + \sigma + \alpha_F - 3\sigma_L \leq \beta Q \tag{16}
\]
\[
\beta Q \leq \frac{10\alpha_L + 13\alpha_F - 2\theta - \sigma}{13} \tag{17}
\]
\[
\beta Q \leq \alpha_F + \frac{4\theta + 2\sigma + 15\alpha_L}{44} \tag{18}
\]

To guarantee that these conditions hold with probability one we evaluate each condition in the most unfavorable set of values of \( \alpha_L \) and \( \alpha_F \) for its fulfillment. After rearranging and simplifying the resulting expressions we conclude that, in this case, there is an interior solution for sure both in the auction and the secondary market if and only if the following condition holds:

\[
2\sigma < \beta Q < \frac{63\theta}{44} + \frac{\sigma}{22} \tag{19}
\]

what implies that the number of permits, weighted by the slope of the marginal abatement cost function cannot be neither too low nor two high, to ensure that both firms will be willing to get at least some permit and the permit are not so abundant that the price could fall to zero or become negative. Note also that, for the interval \( (2\sigma, \frac{63\theta}{44} + \frac{\sigma}{22}) \) to be non-empty, the amplitude of the distribution cannot be two high compared to \( \theta \). Specifically, \( \sigma < \frac{63\theta}{86} \).

Since the expected values of the types is the same, it is immediate to conclude that, when grandfathering is used, the permits are symmetrically shared between firms, i.e., \( \gamma_L = \frac{\mu_L}{\mu_L + \mu_F} = \frac{1}{2} = \gamma_F \).

CASE 2: \( \alpha_L \leq \alpha_F \)

Based on the shape of the regression function \( E\{\alpha_L/\alpha_F\} = \lambda_0 + \lambda_1 \alpha_F \), it is easy to conclude that, in this case, it must be \( \lambda_0 = \frac{\theta}{2}, \lambda_1 = \frac{1}{2} \). Using these values in conditions (11)-(14) and evaluating them in the most unfavorable values of \( \alpha_L \) and \( \alpha_F \) we get the following version for the interior solution condition that has a similar interpretation of that in case 1:

\[
2\sigma < \beta Q < \frac{63\theta}{44} \tag{20}
\]

and the fulfilment of this condition implies \( \sigma < \frac{63\theta}{86} \). Calculating the unconditional mean of \( \alpha_L \) and \( \alpha_F \) in this case, we get \( \bar{\mu}_L = \theta + \frac{\bar{\gamma}}{2}, \bar{\mu}_F = \theta + \frac{2\bar{\gamma}}{2} \) and, therefore, the share of both firms in the distribution of
permits is given by $\gamma_L = \frac{3\theta + \sigma}{\theta + 3\sigma}$, $\gamma_F = \frac{3\theta + 2\sigma}{\theta + 3\sigma}$.

**CASE 3: $\alpha_L \geq \alpha_F$**

Following a similar reasoning to that used above, it is easy to conclude that, in this case, $\lambda_0 = \frac{\theta + \sigma}{2}$, $\lambda_1 = \frac{1}{2}$. After using these values in (11)-(14) and checking for the most unfavorable cases, we conclude that there exists interior solution with probability one if and only if

$$\sigma < \beta Q < \frac{63\theta}{44} + \frac{\sigma}{22} \quad (21)$$

what, in turn, requires and $\sigma < \frac{3\theta}{2}$. In this case, the expected values of $\alpha_L$ and $\alpha_F$ are given by $\bar{\alpha}_L = \theta + \frac{2\sigma}{3}$, $\bar{\alpha}_F = \theta + \frac{\sigma}{3}$ and the resulting permits shares are given by $\gamma_L = \frac{3\theta + 2\sigma}{\theta + 3\sigma}$, $\gamma_F = \frac{3\theta + \sigma}{\theta + 3\sigma}$.

The following proposition shows that, in the conditions described above, the auctioning solution is less cost-effective than the grandfathering solution with probability one.

**Proposition 3** In cases 1, 2 and 3, under the relevant conditions for interior solutions, the total cost of abatement when the permits are auctioned is higher than under grandfathering, i.e., $TC^A > TC^G$ with probability one. ■

The result shown in Proposition 3 is surprisingly powerful since it states that, in any event, auctioning will always be beaten by grandfathering from the point of view of cost. It deserves to be stressed that this is not only true in expected terms, but it is also true for every feasible realization of the types (as far as interior solution is guaranteed). The main message of this result is that, in a framework in which there is market power in the secondary market, introducing an auction in which the leader in the secondary is also so in the auction can only worsen the results in terms of costs as compared to grandfathering. Moreover, when in equilibrium both firms get a positive amount of permits in the auction and the secondary market, this result holds with certainty. The reason is that the leader will have strong incentives to use its leadership to distort the market in its own benefit.

The conclusion that can be drawn from this analysis is that, in a framework in which there is not perfect competition in the secondary market, distributing pollution allowances by means of an auction is likely to worsen the situation (in terms of efficiency) as long as it cannot be avoided that the market power spills over to the auction. To shed some additional light on this result, consider the following corollary.

**Corollary 4** In cases 1, 2 and 3, under the relevant conditions for interior solutions, the amount of permits that the leader receives in equilibrium is lower than the leader’s efficient amount of permits and lower than the amount that he would receive under grandfathering. Moreover, the clearing price of the auction is lower than the price of the secondary market, i.e.,

$$q^A_{L0} < q^*_L, \quad q^A_{L0} < q^G_{L0}, \quad p^A_0 < p^1_1.$$
This corollary clarifies how the auction behaves with respect to the efficient allocation. The leader will always receive less permits from the auction than what would be efficient and, therefore, will act as a monopsonist in the secondary market. Under grandfathering, the leader might receive more or less permits than the efficient amount, depending on the realizations of the types (and, if by chance both types were equal to its average, then grandfathering solution would be efficient) but in the event that it receives less than efficient, it would be closer to the efficient solution than the auctioning solution. What the leader does is to overstate his demand in the primary market to keep the price low. When he goes to the secondary market to buy permits, the price increases to some extent, although it will always lower than the price that would prevail under perfect competition.

Next, we wonder if the situation can be improved if the auction is designed in such a way that the leader can not exert any market power in the auction (although it holds such power in the secondary market).

4 Auction with non-strategic players (work in progress)

In this section we assume that both players, the leader and follower, act non-strategically in the auction, that is, each player neglects at the time of bidding the effect of its own bid schedule on the auction equilibrium price. Still, we keep each player’s role at the secondary market and, consequently, each player at the auction is aware of it.

The rationale under this case is that the planner has a more direct control on the auction since it acts as an auctioneer and sets the rules of the auction. This way, it can watch over the competition and punish abusive behavior, whereas such control is not so direct in the secondary market.

Roughly speaking, the main conclusion is that this setting broadens the room for the auction to outperform the grand-fathering allocation. Our analysis is still preliminary, but it points to the following reasoning. Suppose the random draws of the players types are such that \( \alpha_F \) and \( \alpha_L \) are, respectively, high and low enough. Now, \( F \) has a strong need to get permits and expects post-auction market conditions to be worse than in the auction. Additionally, \( L \) has no such a big need and, more importantly, \( L \) can perhaps predict \( F \)’s need (for the case of correlated types) but also \( L \) perceives that he cannot interfere in \( F \)’s assignment at the auction. Then, there are equilibria in which \( F \) gets all of the permits in the auction, since he bids more aggressively than \( L \) does. Of course, there is a re-assignment in the secondary market at a price chosen by \( L \), but so to say, the secondary market departs from a more advantageous position for the environmentally less efficient firm than it does when it goes from the grand-fathering. A still ongoing analysis reveals that this reasoning is still qualitatively valid for auction equilibria in which \( F \) takes most of the permits in the auction.
Proposition 5 In cases 1,2 and 3, under the relevant conditions for interior solutions at the secondary market, there exists a convex subset, \( \Omega \), in the support of types with non-zero probability, such that, if the pair of types lies in \( \Omega \): (i) there exists an auction equilibrium under which \( F \) takes all of the permits in the auction and the secondary market allocation is interior, (ii) the final allocation satisfies \( TC^A < TC^G \).

Moreover, let \( \phi(\alpha_L) \) denote the maximum of \( \alpha_F \) given \( \alpha_L \) in the support of types. The subset \( \Omega \) is such that: in the cases 1 and 2 the pair \( (\phi(\min \alpha_L), \min \alpha_L) \) belongs to it, in case 3 the subset \( \{ (\phi(\alpha_L), \alpha_L) : \alpha_L \in [\theta, \theta + \sigma] \} \) belongs to it.

In words, the main message of Proposition 5 is that, as long as the auctioneer is capable of preventing market power in the auction, there are good changes that the auction results in more efficient allocations than grandfathering. The reason for this result is rather intuitive. Recall that, when market power exists in the secondary market, the initial distribution of permits is crucial for the cost-effectiveness of the ultimate allocation. The most desirable case is that in which the initial allocation gives to each firm the cost-effective amount, (1). Nevertheless, this solution is not implementable since the planner cannot observe the realized values of the types.

When the permits are distributed based on expected values, the solution is prone to be more efficient the closer the realization of the types are to the mean. If the realized values are very different from the mean, the grandfathering distribution will be necessarily rather different from the efficient one. The reason for this inefficiency is the lack of information that the planner has about the types. Since this information is in the hands of the firms, and the auction gives them the opportunity to interact using this information, the result can be more efficient that under grandfathering.

5 Appendix: proofs

5.1 Proof of Lemma 1

The first order condition of problem (8) is given by:

\[
-\frac{d\tilde{\pi}_L}{dq_{L0}} \cdot \frac{\partial \tau_F}{\partial \tau_0} - [Q - \tau_F(p_0, \alpha_F)] + p_0 \frac{\partial \tau_F}{\partial \tau_0} = 0
\]

which, using the expressions for \( \tilde{\pi}_L \) and \( \tau_F \) and rearranging, can be written as

\[
\frac{2}{5} \tau_F(p_0, \alpha_F) - \frac{8Q}{5} + \frac{3(\alpha_L + 2\alpha_F)}{5\beta} - \frac{9}{5\beta}p_0 = 0.
\]
Using again the expression for $\tau_F$ and expression (6) we get

$$\frac{2\lambda_0}{5\beta} + \frac{(2\lambda_1 + 7)\alpha_F}{5\beta} - \frac{2Q}{5} - \frac{9}{5\beta}p_0 - \frac{4Q}{2\beta} + \frac{3(\alpha_L + 2\alpha_F)}{2\beta} = \frac{9}{2\beta}p_0 = 0$$

which, solving for $p_0$ and simplifying, results in expression (9).

### 5.2 Proof of Proposition 2

Assume the leader’s strategy takes the form $\tau_L(p_0, y) = m_0 + m_1 Q + m_2 \alpha_L + m_3 p_0$. The strategy to proof the proposition is that it is possible to find values for the coefficients $m_0$, $m_1$, $m_2$ and $m_3$ such that the market clearing condition $\tau_F(p_0, \alpha_F) + \tau_L(p_0, \alpha_F) = Q$ has the value of $p_0$ given in (1) as a solution. If this is the case, we conclude that, when $L$ and $F$ bid $\tau_F(p_0, \alpha_F)$ and $\tau_L(p_0, \alpha_F)$ respectively, $p_0^*$ arises as a clearing price for the auction. Using the expressions for $\tau_F(p_0, \alpha_F)$ and $\tau_L(p_0, \alpha_F)$ in the market clearing condition, substituting (1) for $p_0$ and collecting terms, we get the following equation:

$$\frac{2\lambda_0(9 + 2\beta m_3) + 63\beta m_0}{63\beta} + \alpha_F \frac{18\lambda_1 + 9 + \beta m_3(5\lambda_1 + 44)}{63\beta} + Q \frac{54 + 63m_1 - 44\beta m_3}{63} + \beta \frac{63\beta m_2 - 27 + 9\beta m_3}{63\beta} = Q$$

for this equation to hold, we need that the following four conditions meet:

$$\frac{2\lambda_0(9 + 2\beta m_3) + 63\beta m_0}{63\beta} = 0, \quad \frac{18\lambda_1 + 9 + \beta m_3(5\lambda_1 + 44)}{63\beta} = 0, \quad \frac{54 + 63m_1 - 44\beta m_3}{63} = 1, \quad \frac{63\beta m_2 - 27 + 9\beta m_3}{63\beta} = 0,$$

and solving this system of equations, we get the expressions for $m_0$, $m_1$, $m_2$ and $m_3$ shown in the proposition.

### 5.3 Proof of Proposition 3

Using expressions (3) and (10) we get the following expression for the cost difference between auction and grandfathering:

$$TC^A - TC^G = \frac{\beta Q^2 (49\gamma_L - 49\gamma_L^2 - 6)}{441} + \frac{Q [-22\alpha_L + 10\lambda_0 + (12 + 10\lambda_1) \alpha_F - 49\gamma_L (\alpha_F - \alpha_L)]}{441} + \frac{(3\alpha_L - 2\lambda_0 - (2\lambda_1 + 1) \alpha_F) (-4\alpha_L - 2\lambda_0 + (6 - 2\lambda_1) \alpha_F)}{441\beta}.$$
First, note that, in all three cases, \( TC^A - TC^G \) is a continuous and bounded function defined on a compact set and, therefore, we can use the Weierstrass theorem to state that there exists a minimum in the relevant interval. Although the strategy of the proof is the same, the development is slightly different for each case and so we consider them separately.

**CASE 1:** \( \alpha_L \) and \( \alpha_F \) are not correlated. The minimization problem can be written as

\[
\min \Delta(\alpha_L, \alpha_F) \\
\text{s.t.:} \quad \begin{align*}
\theta - \alpha_L & \leq 0 \\
\alpha_L - (\theta + \sigma) & \leq 0 \\
\theta - \alpha_F & \leq 0 \\
\alpha_F - (\theta + \sigma) & \leq 0
\end{align*}
\]

where, for analytical convenience, we have defined \( \Delta(\alpha_L, \alpha_F) \equiv 441/\beta (TC^A - TC^G) \) to eliminate the denominator. The Kuhn-Tucker (K-T) first order necessary conditions for this problem are the following:

\[
\begin{align*}
\frac{\partial \Delta}{\partial \alpha_L} - \mu_1 + \mu_2 &= 0 \\
\frac{\partial \Delta}{\partial \alpha_F} - \mu_3 + \mu_4 &= 0 \\
\mu_1 (\alpha_L - \theta) &= \mu_2 (\alpha_L - \theta - \sigma) = \mu_3 (\alpha_F - \theta) = \mu_4 (\alpha_F - \theta - \sigma) = 0 \\
\mu_1, \mu_2, \mu_3, \mu_4 &\geq 0
\end{align*}
\]

where \( \mu_1, \ldots, \mu_4 \) are the K-T multipliers associated to the constraints. Depending on which constraints are binding, there are 9 possible combinations (1.- No constraint is binding, 2.- Only the first and the third constraints are binding, 3.- Only the first and the fourth are binding, and so on). Exploring each individual case and checking if all the K-T conditions hold, we can discard eight out of nine possibilities and we conclude that the only candidate to being a minimum is \( \alpha_L = \theta, \alpha_F = \theta + \sigma \) (where the first and the fourth constraints are binding and the second and the third are not). Since we know that there is a minimum and we have only one candidate, we can conclude that this is the (global) minimum. By plugging the values of \( \alpha_L \) and \( \alpha_F \) in the expression for \( \Delta \) we get

\[
\Delta = \beta Q (6.25 \beta Q - 7.5 \sigma) - 10 \sigma^2.
\]

Using condition (19) we can find a lower bound for \( \Delta \) by substituting \( 2\sigma \) for \( \beta Q \) and we get \( \Delta > 0 \). Therefore, it is proved that \( TC^A > TC^G \) with probability one in the relevant interval for case 1.

**CASE 2:** \( \alpha_L \leq \alpha_F \). The minimization problem can be written as
\[
\min \Delta (\alpha_L, \alpha_F) \\
\text{s.t.:} \quad \theta - \alpha_L \leq 0 \\
\quad \alpha_F - (\theta + \sigma) \leq 0 \\
\quad \alpha_L - \alpha_F \leq 0.
\]

The Kuhn-Tucker (K-T) first order necessary conditions for this problem are the following:

\[
\begin{align*}
\frac{\partial \Delta}{\partial \alpha_L} - \mu_1 + \mu_3 &= 0 \\
\frac{\partial \Delta}{\partial \alpha_F} + \mu_2 - \mu_3 &= 0 \\
\mu_1 (\alpha_L - \theta) &= \mu_2 (\alpha_F - \theta - \sigma) = \mu_3 (\alpha_L - \alpha_F) = 0 \\
\mu_1, \mu_2, \mu_3 &\geq 0
\end{align*}
\]

and the necessary second order conditions state that \(d' \cdot H \cdot d \geq 0\), \(H\) being the Hessian matrix of \(\Delta\) for any vector \(d \in \mathbb{R}^2\) such that \(g_i \cdot d = 0\), where \(g_i\) is the gradient of those constraints that are binding. In this case we have seven possibilities, of which we can discard six and we conclude that the unique candidate for being a minimum is again \(\alpha_L = \theta, \alpha_F = \theta + \sigma\), that gives the following value for \(\Delta\):

\[
\Delta = \frac{\beta^2 Q^2 (225 (\theta + \sigma) + 44\sigma^2) + \beta Q \sigma (6\theta + 3\sigma) (-45\theta + 2\sigma) - 10 (6\theta + 3\sigma)^2 \sigma^2}{(6\theta + 3\sigma)^2}
\]

after checking that \(\Delta\) increases with \(\beta Q\) and using condition (20) we can find a lower bound for \(\Delta\) by substituting \(2\sigma\) for \(\beta Q\) and we get

\[
\Delta > \frac{(98\sigma^2 + 294\theta\sigma) \sigma^2}{(6\theta + 3\sigma)^2} > 0
\]

and, therefore, \(TC^A > TC^G\) holds with probability one in the relevant interval for case 2.

**CASE 3:** \(\alpha_L \geq \alpha_F\). The minimization problem can be written as

\[
\min \Delta (\alpha_L, \alpha_F) \\
\text{s.t.:} \quad \theta - \alpha_F \leq 0 \\
\quad \alpha_L - (\theta + \sigma) \leq 0 \\
\quad \alpha_F - \alpha_L \leq 0.
\]
The Kuhn-Tucker (K-T) first order necessary conditions for this problem are the following:

\[ \frac{\partial \Delta}{\partial \mu_L} + \mu_2 - \mu_3 = 0 \]
\[ \frac{\partial \Delta}{\partial \mu_R} - \mu_1 + \mu_3 = 0 \]
\[ \mu_1 (\alpha_F - \theta) = \mu_2 (\alpha_L - \theta - \sigma) = \mu_3 (\alpha_F - \alpha_L) = 0 \]
\[ \mu_1, \mu_2, \mu_3 \geq 0 \]

and the necessary second order conditions is analogous to that shown for case 2. There are again seven possibilities and using the first order or second order necessary conditions, we can discard five and we come up with two candidates to be a minimum: \( \alpha_L = \theta + \sigma, \alpha_F = \theta + \sigma \) and \( \alpha_L = \theta + \sigma, \alpha_F = \theta \). By comparing both, we conclude that the former (the latter) is the (global) minimum if \( 15\theta < 17\sigma (15\theta < 17\sigma) \). Anyway, it is easy to check that, in both candidates \( \Delta > 0 \) holds for the relevant interval, what implies that in the minimum of \( \Delta \), and hence everywhere, it holds that \( TC^A > TC^G \). ■

References


